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Estimation under Progressive Type-II censoring for Reciprocal Exponential Distribution

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Abstract

Estimation related to the parameters of reciprocal exponential distribution is discussed for progressively type-II censored samples. A maximum likelihood estimator for the parameters is developed. A simulation study is considered for different pattern of censoring.

Here we have used the simulation algorithm given by Aggarwala (2001) to generate samples. Here we have specified the proportion of surviving units to be removed at five monitoring and censoring point. The percentages of removing units from the surviving units at five stages are increasing in pattern R_1 while decreasing in pattern R_2. In pattern R_3, a conventional type- II censoring scheme is employed.

Using the likelihood level r, the likelihood inequality can be solved in order to construct a likelihood interval for θ. From a graph of likelihood ratio= $L(\hat{\theta})/L(\theta)$ plotted against various values of θ, the likelihood interval for θ can be obtained for given level r, by drawing a horizontal line at $L(\hat{\theta})/L(\theta)$ =r and the corresponding likelihood interval will contain all values of θ below this line. For bootstrapping, we again have simulated 1000 samples using the value of $\hat{\theta}$ as a true

value of θ and calculated $\hat{\theta}_{boot}(p)$ for p = 0.025 and p = 0.975 to obtain the 95% confidence interval.

Keywords: Reciprocal exponential distribution, progressive type-II censoring, maximum likelihood estimation, confidence interval.

1.Introduction

The times to the occurrences of events are termed as "lifetimes". i.e. the actual length of an individual is termed as lifetime. When we buy any item or device such as television, computer, electric bulb etc, we expect it to function properly for a reasonable period of time. i.e. we would like to know the average life or warranty period of an item. Thus reliability function is nothing but the survival function of an item.

In a life testing experiment, items are subjected to test and failed times of items are observed. From practical point of view it is just not possible to examine the sample fully. A complete examination of a sample involves considerable amount of time and money. In addition one requires sufficient space for conducting the experiment. This further adds to the costs of life-test experiment. Hence on account of time and cost consideration a sample has to be truncated. Truncation of the sample is known as censoring.

There are many types of censoring schemes, but type-I and type-II censoring schemes are generally used. If we terminate the experiment when a pre assigned time is observed, such an experiment is known as time censored sampling or Type-I censoring. This kind of censoring is used when cost of experiment increases heavily with time. In type-II censoring a life test is terminated as soon as fixed number if items (say r) have failed. Such an experiment is known as failure censored sampling which is related with very high cost sophisticated items such as color television tubes.

Generally type-I and type-II censoring schemes do not allow removal of units at points other than the terminal point of experiment. A generalized censoring scheme, defined by

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Original Article

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Cohen (1963) which is known as progressive type-II censoring scheme is described below.

Before conducting, a life experiment the experimenter fixes a sample size n, a number of complete observation m and a censoring scheme (R_1, R_2, …… R_m), n = $m + \sum R_i$. The n units are placed on a life test. Immediately after first failure, R_1 surviving units are randomly chosen and removed from the experiment. Then after second failure, R_2 units are withdrawn and so on. The procedure is continued until all R_m remaining units are removed after this m^{th} failure.

If $R_1 = R_2 = \ldots\ldots\ldots\ldots = R_m = 0$, then n = m which corresponds to complete sample. If $R_1 = R_2 = \ldots\ldots\ldots\ldots = R_{m-1} = 0$ them $R_m = n - m$ corresponds to conventional type II right censoring scheme.

Balakrishnan and Aggarwala (2000) has provided a comprehensive reference in the subject of progressive censoring, its application and techniques for analyzing data from the employment of progressive type II censoring schemes.

In this paper we have considered reciprocal exponential distribution as a continuous lifetime model and apply progressively type-II censoring without changing the parameters at different stages of censoring. In section 2 the method of maximum likelihood estimation described. Simulation of progressive type-II censored samples is carried out in section 3. Section 4 deals with confidence interval under three different methods. The methods are illustrated using numerical examples for different censoring pattern.

ISSN: 2147-1428

Original Article

International Journal of Basic and Clinical Studies (IJBCS)

2013;1(1): 212-224. Kaur S et al.

1. Maximum Likelihood Estimation

The probability density function and cumulative distribution function of a reciprocal exponential distribution with parameter θ is given by,

$$g(x) = \frac{\theta}{x^2} e^{\frac{-\theta}{x}}, x > 0,\ \theta > 0 \qquad (2.1)$$

$$G(x) = e^{\frac{-\theta}{x}} \qquad (2.2)$$

Let n items are kept on test, then the likelihood function under Progressive type – II censoring scheme as discussed in section 1 is given by

$$L = \text{constant} \prod_{i=1}^{m} g(x_i/\theta) \prod_{i=1}^{m} \left[1 - G(x_i/\theta)\right]^{R_i}$$

Using (2.1) and (2.2) the likelihood function becomes

$$L = \text{constant} \frac{\theta^m}{\prod_{i=1}^{m} x_i^2} e^{-\theta \sum_{i=1}^{m} \frac{1}{x_i}} \prod_{i=1}^{m} \left(1 - e^{-\theta/x_i}\right)^{R_i} \qquad (2.3)$$

The log likelihood function is given by,

$$\ln L = \ln c + m \ln\theta - \theta \sum_{i=1}^{m} \frac{1}{x_i} - \sum_{i=1}^{m} \ln x_i^2 + \sum_{i=1}^{m} R_i \ln\left(1 - e^{-\theta/x_i}\right)$$

ISSN: 2147-1428

Original Article

International Journal of Basic and Clinical Studies (IJBCS)

2013;1(1): 212-224. Kaur S et al.

Differentiating ln L with respect to θ and equating to zero we obtain,

$$\frac{m}{\theta} - \sum_{i=1}^{m} \frac{1}{x_i} + \sum_{i=1}^{m} \frac{R_i}{1 - e^{-\theta/x_i}} \left(-e^{-\theta/x_i}\right)\left(\frac{-1}{x_i}\right) = 0 \qquad (2.4)$$

Hence we obtain the maximum likelihood estimating equation as,

$$\hat{\theta} = \frac{m}{\sum_{i=1}^{m} \frac{1}{x_i} - \sum_{i=1}^{m} \frac{R_i e^{-\theta/x_i}}{x_i \left(1 - e^{-\theta/x_i}\right)}} \qquad (2.5)$$

Using any iterative procedure like Newton Raphson method one can solve the equation (2.5) to obtain maximum likelihood estimator of θ, denoted by $\hat{\theta}$.

Now again differentiating (2.4) we get,

$$\frac{\partial^2 \ln L}{\partial \theta^2} = \frac{-m}{\theta^2} - \sum_{i=1}^{m} \frac{R_i}{x_i^2} \frac{e^{-\theta/x_i}}{\left(1 - e^{-\theta/x_i}\right)^2} \qquad (2.6)$$

Hence observed asymptotic variance of $\hat{\theta}$ is given by (Due to Cohen 1963)

$$V(\hat{\theta}) = \frac{-1}{\left.\frac{\partial^2 \ln L}{\partial \theta^2}\right|_{\theta=\hat{\theta}}} \qquad (2.7)$$

IJBCS ISSN: 2147-1428

Original Article

International Journal of Basic and Clinical Studies (IJBCS)

2013;1(1): 212-224. Kaur S et al.

2. Comparison of censoring patterns via simulation

In this section considering the reciprocal exponential distribution defined in (2.1) as a life time model from which 1000 samples were generated using the value θ = 3, m = 5, sample size 20 and 50 for each of the following progressive type II censoring patterns

R_1 :(25%, 25%, 50%, 50%, 100%) (ascending)

R_2 :(50%, 50%, 25%, 25%, 100%) (descending)

R_3 :(0, 0, 0, 0, 100%). (regular type II)

Here we have used the simulation algorithm given by Aggarwala (2001) to generate samples. Here we have specified the proportion of surviving units to be removed at five monitoring and censoring point. The percentages of removing units from the surviving units at five stages are increasing in pattern R_1 while decreasing in pattern R_2. In pattern R_3, a conventional type- II censoring scheme is employed.

The simulation scheme is as follows:-

1) Generate U_i, where U_i is a set of random number i= 1, 2, 3, 4, 5

2) $Z_i = -\ln\ (1-U_i)$

3) $$Y_i = \frac{Z_1}{n} + \frac{Z_2}{n-R_1-1} + \ldots\ldots\ldots\ldots\ldots + \frac{Z_i}{n-\sum_{j=1}^{i-1} R_j - i + 1}$$

4) $G(x_i) = 1 - \exp(-Y_i)$

i.e. $\exp(-\theta/x_i) = 1 - \exp(-Y_i)$

IJBCS ISSN: 2147-1428

Original Article

International Journal of Basic and Clinical Studies (IJBCS)
2013;1(1): 212-224. Kaur S et al.

By solving the equation in step 4 we will get the values of x_i. On the basis of simulated samples Maximum Likelihood estimates of θ as given in (2.5) along with its asymptotic variance as given in (2.7) with simulated variance are demonstrated in table 1.

For n = 20 the three censoring patterns as discussed earlier the comes out as

R_1 :(5, 3, 5, 2, 0)

R_2 :(10, 4, 1, 0, 0)

R_3 :(0, 0, 0, 0, 15).

Table-1 gives the summary statistics of the maximum likelihood estimators for the three censoring patterns; with its observed asymptotic variance and simulated variance, in case of 1000 random samples generated for n = 20, θ = 3, m= 5

Table-1

Scheme	**Min** $\hat{\theta}$	**Max** $\hat{\theta}$	$\hat{\theta}$	Asy $V(\hat{\theta})$	Sim $V(\hat{\theta})$
R_1	1.5945	7.2547	3.3064	0.8109	0.8631
R_2	1.3139	9.481	3.4278	1.0163	1.0735
R_3	1.6171	7.7873	3.3125	0.6656	0.6236

From the result of table 1 we observe that the censoring pattern R_3 produces the most precise estimate of θ followed by R_1 and then R_2. This is due to the fact that more units are kept in the experiment for a longer period of time in R_3 followed by R_1 and then R_2.

IJBCS ISSN: 2147-1428

Original Article

International Journal of Basic and Clinical Studies (IJBCS)

2013;1(1): 212-224. Kaur S et al.

For n = 50 the three censoring patterns as discussed earlier comes out as

R_1 :(12, 9, 13, 6, 5)

R_2 :(25, 12, 2, 2, 4)

R_3 :(0, 0, 0, 0, 45).

Table-2 gives the summary statistics of the maximum likelihood estimators for the three censoring patterns; with its observed asymptotic variance and simulated variance, in case of 1000 random samples generated for n = 50, θ = 3, m= 5

Table - 2

Scheme	**Min** $\hat{\theta}$	**Max** $\hat{\theta}$	$\hat{\theta}$	Asy $V(\hat{\theta})$	Sim $V(\hat{\theta})$
R_1	1.5742	5.5017	3.0504	0.3436	0.3259
R_2	1.3285	5.7127	3.0504	0.4069	0.4103
R_3	1.6867	4.9825	3.0386	0.2837	0.2702

From the result of table - 2 we observe that the censoring pattern R_3 produces the most precise estimate of θ followed by R_1 and then R_2. This is due to the fact that more units are kept in the experiment for a longer period of time in R_3 followed by R_1 and then R_2. This shows that result obtained for small sample is same as the one obtained with large sample.

3. Confidence Interval Estimation

In this section we consider interval estimation of unknown parameter θ using the method of parametric bootstrap confidence interval and the method of r-level likelihood.

Original Article
International Journal of Basic and Clinical Studies (IJBCS)
2013;1(1): 212-224. Kaur S et al.

According to Davison and Hinkley (1997) a 100(1-α) % parametric bootstrap confidence interval for θ is given by

$$\left(\frac{\hat{\theta}^2}{\hat{\theta}_{boot}(1-\alpha/2)}, \frac{\hat{\theta}^2}{\hat{\theta}_{boot}(\alpha/2)}\right) \qquad (4.1)$$

where $\hat{\theta}_{boot}(p)$ is the p[th] percentile of the simulated sample of 1000 estimates simulated using the observed value of $\hat{\theta}$ of the given sample.

Using the likelihood level r, the likelihood inequality can be solved in order to construct a likelihood interval for θ. From a graph of likelihood ratio= $L(\hat{\theta})/L(\theta)$ plotted against various values of θ, the likelihood interval for θ can be obtained for given level r, by drawing a horizontal line at $L(\hat{\theta})/L(\theta)$=r and the corresponding likelihood interval will contain all values of θ below this line. For bootstrapping, we again have simulated 1000 samples using the value of $\hat{\theta}$ as a true value of θ and calculated $\hat{\theta}_{boot}(p)$ for p = 0.025 and p = 0.975 to obtain the 95% confidence interval in case of all the three censoring patterns and the values are as follows:

	R_1	R_2	R_3
$\hat{\theta}_{boot}(0.025)$	1.8607	1.9545	2.077
$\hat{\theta}_{boot}(0.975)$	5.456	5.872	5.0326

Using the result given in (4.1), parametric bootstrap confidence interval for θ in case of all the three censoring patterns R_1, R_2 and R_3 is given by (1.269692, 4.849047), (1.414528, 5.635088), (1.042647, 4.394512) respectively whereas likelihood level r = 5 confidence intervals of θ for the schemes R_1, R_2 and R_3 are obtained as (1.7095, 3.23449),

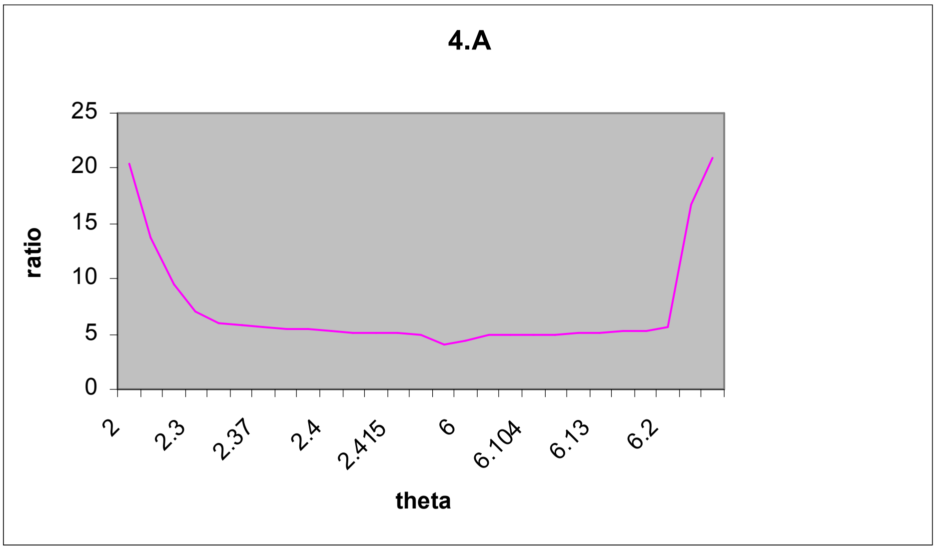
IJBCS ISSN: 2147-1428

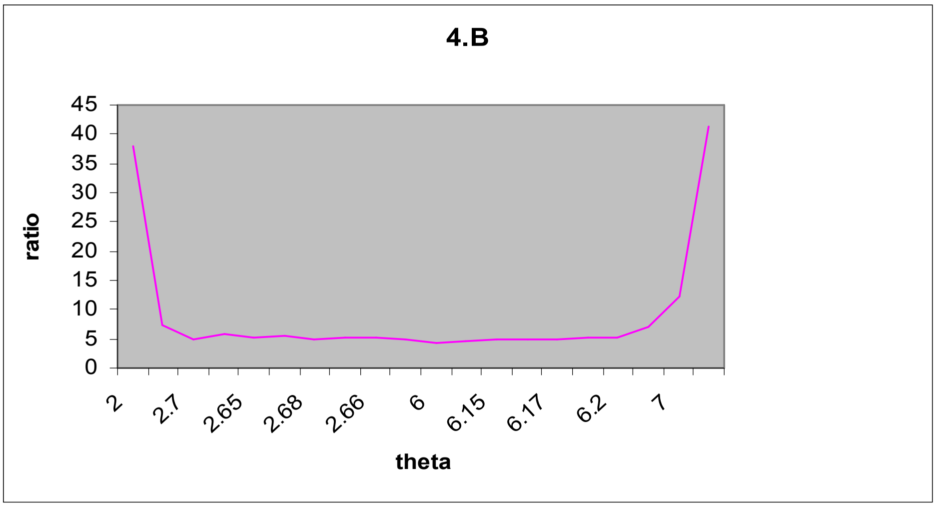
Original Article

International Journal of Basic and Clinical Studies (IJBCS)

2013;1(1): 212-224. Kaur S et al.

1.7842, 4.7749) and (2.14599,5.1862) respectively. The graph of ratio= $L(\hat{\theta})/L(\theta)$ versus θ are shown in the following three figures 4.A, 4.B and 4.C respectively for censoring patterns R_1, R_2 and R_3. The advantage of likelihood level confidence interval estimation is that it does not require large amounts of simulation as required in bootstrapping.



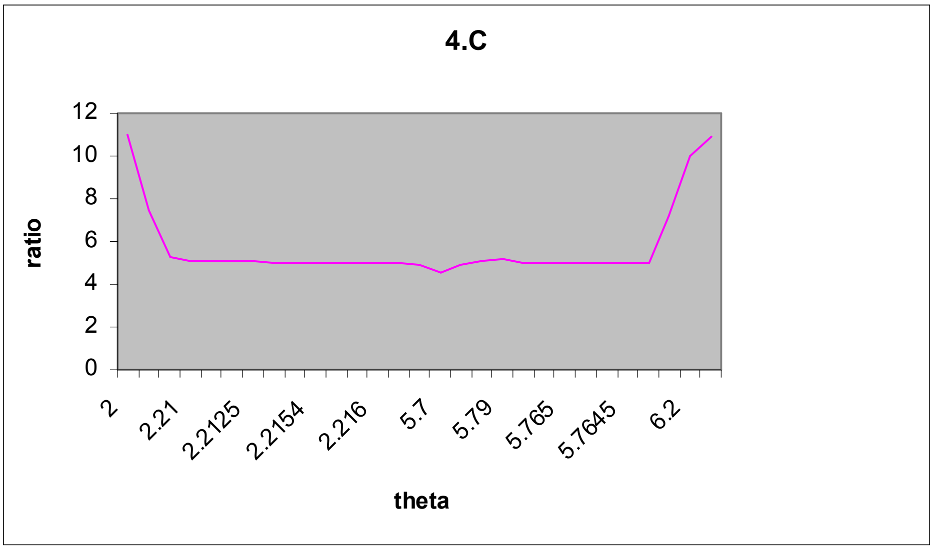


IJBCS ISSN: 2147-1428

Original Article

International Journal of Basic and Clinical Studies (IJBCS)

2013;1(1): 212-224. Kaur S et al.



Now instead of taking 1000 samples we took single sample and computed the values for different values of likelihood level for each censoring pattern to check the effect it creates on the confidence interval. We found that as the likelihood level is increased the confidence interval also increases for the three censoring patterns, i.e. the difference between the lower limit and upper limit increases. The result is shown in the table given below.

For n = 20, $\theta = 3$, m = 5

Likelihood level (r)	R_1	R_2	R_3
3	(2.63635, 5.66161) Diff : 3.02526	(2.72244, 6.0743) Diff : 3.35186	(2.50933, 5.1987) Diff : 2.68937
5	(2.41711, 6.102605) Diff : 3.685495	(2.48913, 6.5807) Diff : 4.09157	(2.30711, 5.57799) Diff : 3.27088
7	(2.29828, 6.36793) Diff : 4.06965	(2.36323, 6.88705) Diff : 4.52382	(2.197025, 5.805) Diff : 3.607975

IJBCS ISSN: 2147-1428

Original Article

International Journal of Basic and Clinical Studies (IJBCS)

2013;1(1): 212-224. Kaur S et al.

Sprott (1973) has indicated that the distribution $\hat{\phi} = \hat{\theta}^{-1/3}$ in small samples is much more closely approximated by a normal distribution than the distribution of $\hat{\theta}$. The distribution of $\hat{\phi}$ is approximately normal with mean $\phi = \theta^{-1/3}$ and variance

$$\mathrm{V}(\hat{\phi}) = \left(\frac{d\phi}{d\theta}\right)^2 .Asy(\hat{\theta})$$

Thus $$\frac{\hat{\phi}-\phi}{\sqrt{\mathrm{V}(\hat{\phi})\big|_{\theta=\hat{\theta}}}} \sim N(0,1) \qquad (4.2)$$

	$\hat{\Phi} = \hat{\theta}^{-1/3}$	$\mathrm{V}(\hat{\theta}) = \frac{1}{9}\hat{\theta}^{-8/3} \times \mathrm{AsyV}(\hat{\theta})$	$1.96 \times \mathrm{V}(\hat{\theta})$
R_1	0.6712	0.0037	0.0073
R_2	0.6632	0.0042	0.0082
R_3	0.6708	0.0030	0.0059

Using the result given in (4.2), confidence interval for θ (given by Sprott) in case of all the three censoring patterns R_1, R_2 and R_3 is given by (1.1404, 1.1463), (1.1420, 1.1515), and (1.1390, 1.1457) respectively.

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